Linear Programming

CMSC 641 Algorithms

A Simple Example

Chocolatier w/ 2 types of chocolate

x_1 = boxes of "Pyramide" per day

$1 profit per box of "Pyramide"

x_2 = boxes of "Nuit"

$6 profit per box of "Nuit"

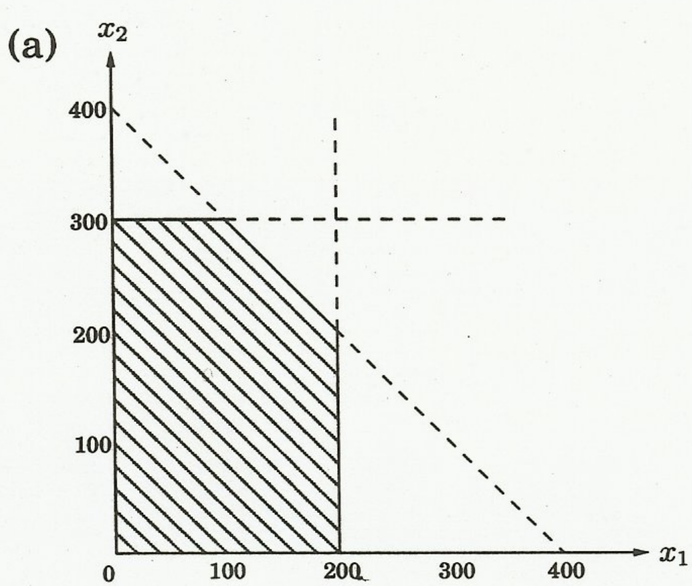
won't sell more than 200 boxes of Pyramide

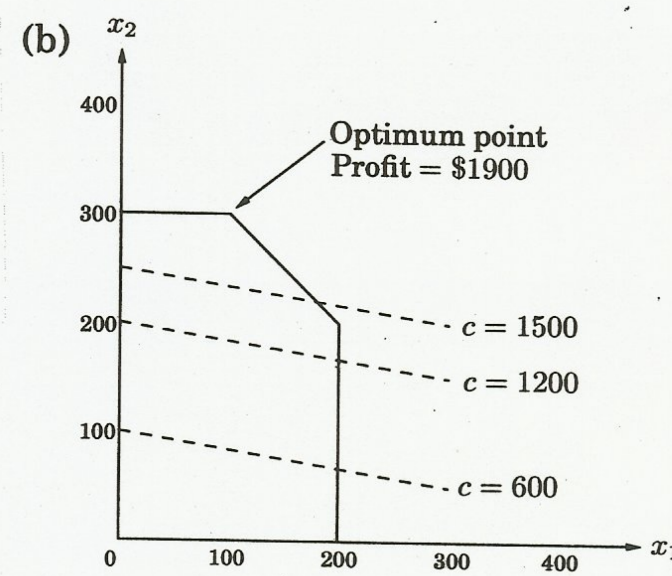
won't sell more than 300 boxes of Nuit

can't make more than 400 boxes

$$\max \quad x_1 + 6x_2 \quad \longleftarrow \text{objective function}$$

$$\left.\begin{aligned} x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}\right\} \quad \text{Constraints}$$





opt solution

$x_1 = 100$

$x_2 = 300$

Add a third type of chocolate

$13 profit per box

additional constraint $x_2 + 3x_3 \leq 600$

$$\max \quad x_1 + 6x_2 + 13x_3$$

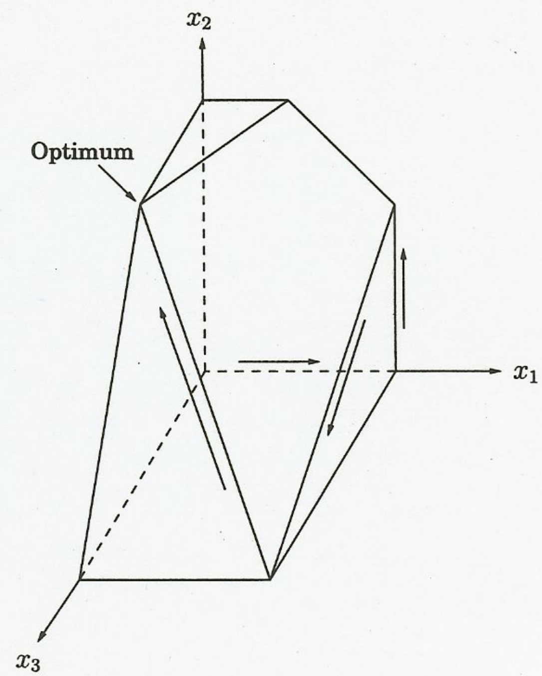
$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 + x_3 \leq 400$$

$$x_2 + 3x_3 \leq 600$$

$$x_1, x_2 x_3 \geq 0$$



opt. solution:

$x_1 = 0$

$x_2 = 300$

$x_3 = 100$

"Duality" - proof of optimality

$$x_2 \le 300$$

$$x_1 + x_2 + x_3 \le 400$$

$$+ \quad 4(x_2 + 3x_3) \le 4 \cdot 600$$

$$\overline{x_1 + 6x_2 + 13 \cdot x_3 \le 3100}$$

We can always prove optimality this way!

Using L.P. to solve graph problems

① Single-source shortest path with positive weight edges (Dijkstra's)

target node = t

Maximize $d[t]$

subject to

$$d[v] \leq d[u] + w(u,v) \quad \forall (u,v) \in E$$

$$d[s] = 0$$

② Maximum Flow

maximize $\sum_{v \in V} f(s,v)$

subject to

$$f(u,v) \leq c(u,v) \quad \forall u,v \in V$$

$$f(u,v) = -f(v,u) \quad \forall u,v \in V$$

$$\sum_{v \in V} f(u,v) = 0 \quad \forall u \in V - \{s,t\}$$

3 Min-cost flow

each edge has a cost $a(u,v)$ per unit

target flow $= d$

Send d units from s to t with least cost

constant

$$\text{minimize} \sum_{(u,v)\in E} a(u,v) \cdot f(u,v)$$

variable

subject to

$$f(u,v) \le c(u,v) \quad \forall\, u,v \in V$$

$$f(u,v) = -f(v,u) \quad \forall\, u,v \in V$$

$$\sum_{v\in V} f(u,v) = 0 \qquad \forall\, u \in V - \{s,t\}$$

$$\sum_{v\in V} f(s,v) = d$$

4 Multi-commodity flow
(see textbook)

5 ...

Algorithms for Linear Programming

Simplex method

- most common
- not polynomial time
- Dantzig 1947

Ellipsoid Method

- Khachiyan 1979
- polynomial time, but slow

before that L.P. could have been NP-complete

Karmarkar

- interior point, 1983
- competitive with simplex for large problems
- "Industrial" strength problems have tens of thousands of variables.

Implementing numerical algorithms tricky
stability & robustness concerns.

Converting L.P.'s to "Standard" form

In ***standard form***, we are given n real numbers $c_1, c_2, \ldots, c_n$; m real numbers $b_1, b_2, \ldots, b_m$; and mn real numbers a_{ij} for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. We wish to find n real numbers $x_1, x_2, \ldots, x_n$ that

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \quad \text{for } i = 1, 2, \ldots, m \\ x_j &\geq 0 \quad \text{for } j = 1, 2, \ldots, n\,. \end{aligned}$$

Original L.P. might:

1. minimize objective function
2. allow variables to be negative
3. have equality constraints
4. have $\geq$ constraints other than non-negative constraints

Original L.P.

$$\begin{array}{lrcrcl} \text{minimize} & -2x_1 & + & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0\,, \end{array}$$

Negate coefficients of objective function:

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0\,. \end{array}$$

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2 \\
\text{subject to} & \\
& x_1 + x_2 = 7 \\
& x_1 - 2x_2 \le 4 \\
& x_1 \ge 0 .
\end{array}
$$

Replace variable x_2, allowed to be negative, with $x_2' - x_2''$.

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - 3x_2' + 3x_2'' \\
\text{subject to} & \\
& x_1 + x_2' - x_2'' = 7 \\
& x_1 - 2x_2' + 2x_2'' \le 4 \\
& x_1, x_2', x_2'' \ge 0 .
\end{array}
$$

Replace = by ≤ and ≥

$$\begin{array}{lrcrcrcl}
\text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2' & - & x_2'' & \leq & 7 \\
& x_1 & + & x_2' & - & x_2'' & \geq & 7 \\
& x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\
& & & x_1, x_2', x_2'' & & & \geq & 0 .
\end{array}$$

Multiply thru by -1 to convert $\geq$ to $\leq$

$$
\begin{array}{lrcrcrcr}
\textbf{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0\,.
\end{array}
$$

Rewritten (and equivalent) L.P.
now in standard form

Standard form better for thinking about L.P. as intersection of half-spaces.

"Slack" form better for simplex method.

Slack form uses equality constraints.

Replace:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

By:

$$s = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

$$s \geq 0$$

Name slack variables s as $x_{n+1}, x_{n+2}, \ldots$

L.P. in slack form:

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 .
\end{array}
$$

$x_1, x_2, \ldots, x_6 \geq 0$ implicit

$z =$ objective function

Simplex Method

- optimum solution at some vertex
- find initial feasible solution at some vertex
- move to adjacent vertex if this improves the objective function.
- if all neighboring vertices all have smaller value, optimum solution found.

Simplex Method Example

Original L.P.

$$\begin{array}{llllllll}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
& 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
& & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}$$

Convert to Slack Form

$$
\begin{aligned}
z &= & & 3x_1 &+& x_2 &+& 2x_3 \\
x_4 &= 30 &-& x_1 &-& x_2 &-& 3x_3 \\
x_5 &= 24 &-& 2x_1 &-& 2x_2 &-& 5x_3 \\
x_6 &= 36 &-& 4x_1 &-& x_2 &-& 2x_3 .
\end{aligned}
$$

Basic variables

Non-basic variables

Basic solution: set $x_1 = x_2 = x_3 = 0$, $z = 0$

What if we increase x_1?

z goes up, but x_4, x_5 & x_6 might become negative

x_6 turns negative first. Make x_1 basic, x_6 nonbasic

Swapping x_1 & x_6

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

Now x_1 is basic & x_6 non-basic.

Can't have x_1 on r.h.s. of other constraints.

Substitute:

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$= 30 - \left(9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{6}\right) - x_2 - 3x_3$$

$$= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$

New L.P. (w/ same feasible solutions)

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
\text{basic vars}\left\{
\begin{aligned}
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} .
\end{aligned}
\right.
\end{aligned}
$$

non basic vars

Again consider basic sol'n where non basic vars set to 0. $z = 27$

What if we increase x_3? x_5 first to become negative.

Swapping x_3 & x_5

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

Substitute x_3 in other constraints and also in the objective function:

$$\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}.
\end{aligned}$$

Now, try increasing x_2.

End up swapping x_2 & x_3.

$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2},
\end{aligned}
$$

Now we cannot increase z by increasing the non-basic variable.

We must have optimum solution, $z = 28$ when $x_1 = 8$, $x_2 = 4$, $x_3 = 0$

Duality, revisited

Consider the original L.P.

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
① & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
② & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
③ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}
$$

We want to prove 28 is the optimum solution.

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
1 & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
2 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
3 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}
$$

We want to prove 28 is the optimum solution.

Take $\frac{1}{6} \times 2 + \frac{2}{3} \times 3$:

$$
\begin{array}{r}
\frac{1}{3} x_1 + \frac{1}{3} x_2 + \frac{5}{6} x_3 \leq 4 \\
+ \quad \frac{8}{3} x_1 + \frac{2}{3} x_2 + \frac{4}{3} x_3 \leq 24 \\
\hline
\frac{9}{3} x_1 + \frac{3}{3} x_2 + \frac{13}{6} x_3 \leq 28
\end{array}
$$

$$\tfrac{9}{3}x_1 + \tfrac{3}{3}x_2 + \tfrac{13}{6}x_3 \le 28$$

$$\Rightarrow \quad 3x_1 + x_2 + \tfrac{13}{6}x_3 \le 28$$

$$\Rightarrow \quad 3x_1 + x_2 + 2x_3 \le 28 \qquad \text{since } x_3 \ge 0.$$

$\therefore$ 28 is an optimum solution.

But how did we get $\tfrac{1}{6}$ and $\tfrac{2}{3}$?

We solve another L.P.

Original L.P.

$$\begin{array}{lrcrcrcl}
\text{maximize} & 3x_1 & + & x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
① & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
② & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
③ & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & & & x_1, x_2, x_3 & & & \geq & 0 .
\end{array}$$

We want to multiply each constraint by a coefficient and add the resulting inequalities

Call these coefficients y_1, y_2, y_3.

It would be great if

$$y_1 \cdot 1 + y_2 \cdot 2 + y_3 \cdot 4 = 3$$

$$y_1 \cdot 1 + y_2 \cdot 2 + y_3 \cdot 1 = 1$$

$$y_1 \cdot 3 + y_2 \cdot 5 + y_3 \cdot 2 = 2$$

We want to multiply each constraint by a coefficient and add the resulting inequalities

Call these coefficients y_1, y_2, y_3.

It would be great if

$$y_1 \cdot 1 + y_2 \cdot 2 + y_3 \cdot 4 = 3$$

$$y_1 \cdot 1 + y_2 \cdot 2 + y_3 \cdot 1 = 1$$

$$y_1 \cdot 3 + y_2 \cdot 5 + y_3 \cdot 2 = 2$$

But, x_1, x_2 & $x_3 \geq 0$, so $\geq$ suffices:

$$y_1 + 2y_2 + 4y_3 \geq 3$$
$$y_1 + 2y_2 + y_3 \geq 1$$
$$3y_1 + 5y_2 + 2y_3 \geq 2$$

Also, we want the r.h.s. of $y_1 \times (1) + y_2 \times (2) + y_3 \times (3)$ to be as small as possible. So

minimize $\quad 30y_1 + 24y_2 + 36y_3$

Bonus!

The "primal" L.P. & the "dual" L.P. are so tightly connected that solving the primal gives a solution to the dual... and provides a proof of optimality.

Proofs w/ calculations in the textbook.

Analogous to Max Flow / Min Cut Theorem.

For more "Primal Dual" Algorithms, see Combinatorial Optimization, Papadimitriou & Steiglitz, Dover.

A Primal Duel